International Journal of Scientific & Engineering Research, Volume 4, Issue 11, November-2013
ISSN 2229-5518

Chaotic Behaviour in Some Discrete –Time Adaptive Control Systems

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Abstract— It has been shown that nonlinear discrete maps can display extremely rich behaviour and under certain parameter conditions to show chaotic phenomenon. This work looks at adaptive control feedback systems which can be represented as nonlinear discrete maps and shows how model mismatch can lead to undesired complicated and chaotic behaviour. Moreover that a discrete-time adaptive control system which can display chaotic behaviour can be extended into higher order systems and the results show that under certain parameter conditions, the higher order systems also behave chaotically. A generalised equation form for the eigenvalues is also given.

Index Terms— Adaptive Feedback Control Systems, Bifurcation, Chaos, Nonlinear Mappings, Simulation

1 Introduction

Chaos theory looks at the study of deterministic dynamical systems that are very sensitive to initial conditions. Small differences in initial conditions can lead to widely diverging outcomes, for such systems making long term predictions generally becomes impossible. Chaotic phenomena have been observed in numerous systems in the science and engineering fields [1]. Lorenz [2] made early studies in the changes in the atmosphere which tended to display erratic and unpredictable behaviour. In more recent years, with potential application in engineering fields the study and control of chaotic systems has become important specifically chaos control and synchronization [3].

Adaptive feedback control starts with a system model with known or unknown parameters. Parameter adaptive control looks at global behaviour and a set of parameters which are manipulated by the observer. The observed behaviour is compared with the desired one and corrections made using the system parameters. It is know that adaptive control inherently leads to a nonlinear closed-loop system, even in the situation where we have a linear plant or a linear model of the plant, when a linear output feedback law is used and where the feedback parameters are estimated from input and output data. Adaptive systems can be thought of as being asymptotically linear when the feedback parameters have converged to some steady state. The study considers what sort of dynamical effects on the performance of the adaptive system the nonlinear features can have

A discrete adaptive system can be represented as a nonlinear discrete map. It has been shown that a one-dimensional nonlinear map [4], [5], a one-dimensional map with a quadratic nonlinearity known as the logistic map [6], exhibit unpredictable and chaotic phenomena. Ydstie [7] has shown how a simple model-reference adaptive system (MRAC) characterised by a third order nonlinear discrete map with no external forcing can exhibit chaotic behaviour. More recently, due the potential applications in a variety of disciples, chaos control has become an important consideration like the area of secure communications [8].

In this paper a general n^{th} order plant is taken with increasing order model assumptions and using analysis and simulation show that complex dynamics and chaotic behaviour can occur for higher order systems for certain parameter values. A general closed form equation is developed giving the eigenvalue structure for the nth order plant with a first order model assumption. This provides insight into the unstable behaviour of our systems.

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2 General Background

In model–reference adaptive control, the basic idea is to compare the behaviour of the controlled plant with that of reference model representing the desired performance and attempt to reduce the difference between them by changing the controller parameters in an appropriate manner. The basic structure is as shown in fig 1.

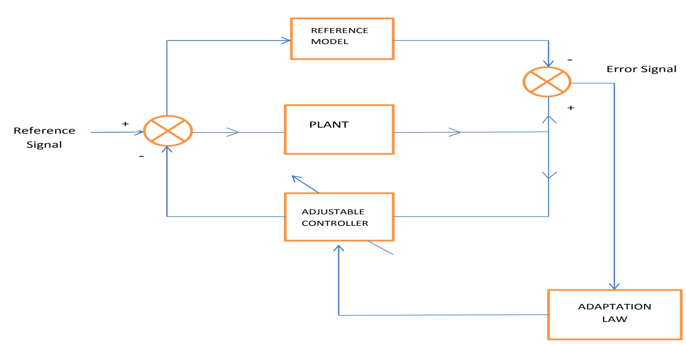

Fig. 1 Model – reference adaptive control system structure.

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The general adaptive control system is designed by combining a particular estimation technique with a control law.

$$y(t) = \alpha(q^{-1})y(t-1) + \beta(q^{-1})u(t-1)$$

$$\alpha(q^{-1}) = a_1 + a_2 q^{-1} + \dots\dots + a_n q^{-(n-1)}$$

$$\beta(q^{-1}) = \beta_1 + \beta_2 q^{-1} + \dots\dots + \beta_m q^{-(m-1)}$$

$$q^{-1}y(t) \triangleq y(t-1)$$

For simplicity $\beta(q^{-1}) = 1$ and $(a_1, .. a_n, u(t), y(t) \in \mathbb{R})$

It is assumed that the system can be adequately represented as a first-order model,

$$y(t) = \hat{a}_1(t-1)y(t-1) + u(t-1)$$

where $\hat{a}_1$ is an estimate of a_1.

The desired output $y(t)$ is be equal to some reference value y^*, we can choose $u(t)$ using an adaptive control law to achieve closed-loop stability and to asymptotically achieve zero tracking error.

$$u(t) = y^* - \hat{a}_1(t)y(t)$$

An algorithm we use to give $\hat{a}_1$, an estimate of the actual parameter a_1.

$$\hat{a}_1(t) = \hat{a}_1(t-1) + py(t-1)e(t)$$

Where p is the adaptation rate and $e(t)$ denotes model error.

3 General nth Order Plant, First Order Model

Plant: $y(t) = \sum_{i=1}^{n} a_i y(t-i) + u(t-1)$

Model: $y(t) = \hat{a}_1(t-1)y(t-1) + u(t-1)$

Control Law: $u(t) = y^* - \hat{a}_1(t)y(t)$

Estimation Algorithm:

$$\hat{a}_1(t) = \hat{a}_1(t-1) + py(t-1)[\, y(t) - \hat{a}_1(t-1)y(t-1) - u(t-1)]$$

This produces a sequence of parameters

$\{ \hat{a}_1(t)\}\ t \geq 1$ with controls $\{ u(t)\}\ t \geq 1$, given $\hat{a}_1(0)$.

Closed-loop system equations become:

$$y(t) = \sum_{i=1}^{n} a_i y(t-i) + y^* - \hat{a}_1(t-1)y(t-1)$$

$$\hat{a}_1(t) = \hat{a}_1(t-1) + py(t-1)[\, y(t) - y^* \,]$$

In order to facilitate analysis and simulation we can rewrite the above equations as a set of (n+1) first order equations:

Let $w_1(t) = w_2(t-1)$

$$w_2(t) = w_3(t-1)$$

$$\vdots$$

$$w_{n-2}(t) = w_{n-1}(t-1)$$

$$w_{n-1}(t) = y(t-1)$$

So,

$$y(t) = [a_1 - \hat{a}_1(t-1)]y(t-1) + \sum_{i=1}^{n-1} a_{n+1-i} w_i(t-i) + y^*$$

$$\hat{a}_1(t) = \hat{a}_1(t-1) + py(t-1)[\, y(t) - y^* \,]$$

4 Equilibrium Points

Replacing t by $t-1$ in the left hand side equations gives:

$$w_i(t-1) = y(t-1)$$
$$y(t) = y^*$$
$$\hat{a}_1 = \sum_{i=1}^{n} a_i$$

From which is it can be seen that if there is no plant to model mismatch (i.e. n=1) then our parameter estimate $\hat{a}_1(t)$ at equilibrium is equal to a_1.

5 Eigenstructure for the General Model

The set of (n+1) first order equations represents a nonlinear discrete mapping. The stability of the adaptive system can be determined by looking at the eigenvalue structure of the Jacobian matrix at the fixed point.

Linearization of the system equations at the equilibrium point gives the Jacobian of first partial derivatives DF_θ:

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$$\begin{bmatrix} \Delta w_1(t) \\ \Delta w_2(t) \\ \vdots \\ \vdots \\ \vdots \\ \Delta w_{n-1}(t) \\ \Delta y(t) \\ \Delta \hat{a}_1(t) \end{bmatrix} = DF_\theta \begin{bmatrix} \Delta w_1(t-1) \\ \Delta w_2(t-1) \\ \vdots \\ \vdots \\ \vdots \\ \Delta w_{n-1}(t-1) \\ \Delta y(t-1) \\ \Delta \hat{a}_1(t-1) \end{bmatrix}$$

where, the matrix of first partial derivatives DF_θ is given by :

$DF_\theta =$

$$\begin{bmatrix} 0 & 1 & 0 \cdots\cdots\cdots & 0 & 0 & 0 \\ 0 & 0 & 1 \cdots & \vdots & & 0 \\ \vdots & & & & & \vdots \\ \vdots & & & \vdots & & \vdots \\ \vdots & & & 0 & 1 & 0 \\ a_n & a_{n-1} & \ldots & \ldots\ a_2 & a_1 - \hat{a}_1 & -y \\ a_n py & a_{n-1}py & \ldots & a_2 py & (a_1 - \hat{a}_1)py & 1 - py^2 \end{bmatrix}$$

at $w_i = y = y^*, \ \hat{a}_1 = \ \sum_{i=1}^{n} a_i$

The characteristic equation $\chi(z)$ for DF_θ is given by

$\chi(z) = \det(DF_\theta - zI)$

$\chi(z) =$

$$\begin{vmatrix} -z & 1 & 0 \cdots\cdots & 0 & 0 & 0 \\ 0 & -z & 1 \cdots & 0 & 0 & 0 \\ 0 & 0 & -z & \vdots & \vdots & \vdots \\ \vdots & & & 1 & 0 & 0 \\ \vdots & & & -z & 1 & 0 \\ a_n & a_{n-1} & \ldots & a_2 & a_1 - \hat{a}_1 - z & -y^* \\ a_n py^* & a_{n-1}py^* & \ldots & a_2 py^* & (a_1 - \hat{a}_1)py^* & 1 - py^{*2} - z \end{vmatrix}$$

A concise algebraic form for $\chi(z)$ is:

$$\chi(z) = (1-z)\left[(-1)^n\left(z^n + \hat{a}_1 z^{n-1} - \sum_{r=1}^{n} a_r z^{n-r}\right)\right] + py^{*2}(-1)^{n-1} z^n$$

So, $\chi(z) = 0$ is a polynomial equation giving the eigenvalues for a general n[th] order plant with a first order model assumption.

6 Analysis and Simulation Results

The simplest case in our general representation model with n=2 has been shown to exhibit chaotic behaviour under certain parameter conditions, Ydstie [3]. We consider a plant which is now of third order assuming it can be modelled by a first order system (i.e. n=3).

6.1 Third Order Plant with First Order Model

Our system equations become:

$$w_1(t) = w_2(t-1)$$
$$w_2(t) = y(t-1)$$
$$y(t) = a_1 y(t-1) + a_2 w_2(t-1) + a_3 w_1(t-1) + y^* - \hat{a}_1(t-1)y(t-1)$$
$$\hat{a}_1(t) = \hat{a}_1(t-1) + py(t-1)[\ y(t) - y^*\ ]$$

The above equations can be represented as a nonlinear discrete mapping $F_\theta(w_1, w_2, y, \hat{a}_1) : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ where $(w_1, w_2, y, \hat{a}_1)$ is the state vector, $\theta = (a, a_2, a_3, p, y^*)$ is a vector of parameters.

The eigenstructure for this system is given by the characteristic equation, $\chi(z) = 0$ with n=3,

$$(z-1)(\ z^3 - (a_1 - \hat{a}_1)z^2 - a_2 z - a_3\ ) + py^{*2} z^3 = 0$$

By varying different parameters of θ, we can observe the system's dynamical behaviour. Letting parameter a_2 vary over a range of values from 0.2 to 0.8, with

$$\begin{bmatrix} w_1(0) \\ w_2(0) \\ y(0) \\ \hat{a}_1(0) \end{bmatrix} = \begin{bmatrix} 0.1 \\ 1.0 \\ 1.0 \\ 0.5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ p \\ y^* \end{bmatrix} = \begin{bmatrix} 0.6 \\ a_2 \\ 0.01 \\ 0.005 \\ 5.0 \end{bmatrix}$$

Fig. 2 shows $y(i)$ as a_2 ranges from 0.2 - 0.8.

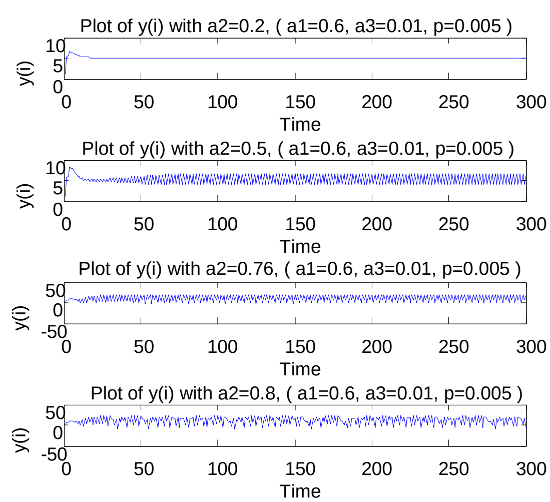

Fig. 2 – Plot of $y(i)$

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ISSN 2229-5518

Fig. 3 below shows $y(i)$ chaotic for 2000 iterations.

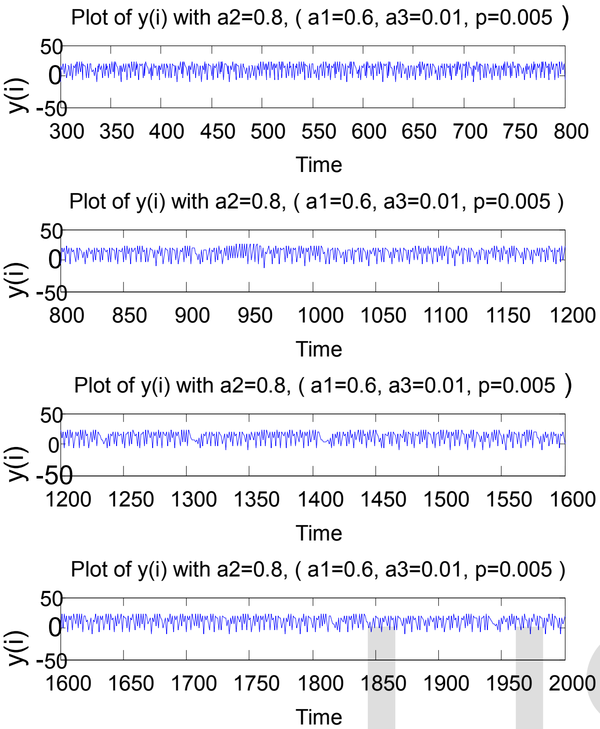

Fig. 3 - Plot of $y(i)$

Fig. 4 below shows the corresponding phase-plane plot of y(i) and a(i).

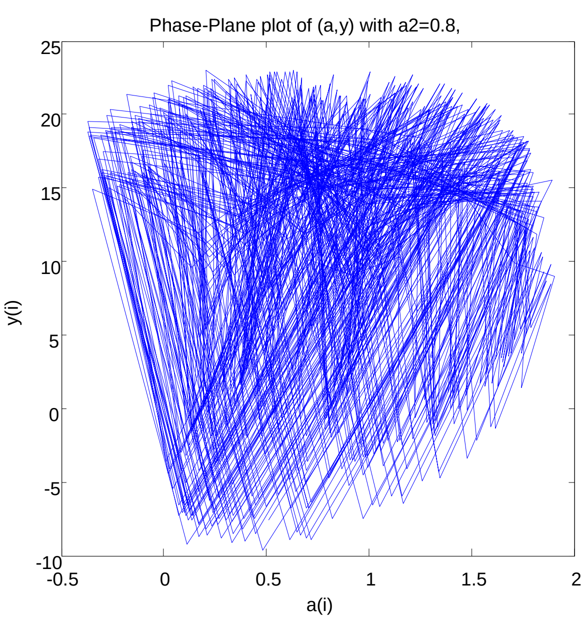

Fig. 4 - Plot of $y(i)$ vs a(i)

It is also seen that as a_2 increases from 0.2 to 0.8 and the system becomes more unstable and eventually chaotic, that one of the eigenvalues becomes greater than one in modulus.

Parameter a_2 with corresponding eigenvalues:
$a_2 = 0.2$ $a_2 = 0.5$ $a_2 = 0.76$ $a_2 = 0.8$

$$\begin{bmatrix} -0.51 \\ -0.05 \\ 0.42 \\ 0.87 \end{bmatrix} \quad \begin{bmatrix} -1.04 \\ 0.57 \\ -0.02 \\ 0.86 \end{bmatrix} \quad \begin{bmatrix} -1.39 \\ 0.65 \\ -0.01 \\ 0.85 \end{bmatrix} \quad \begin{bmatrix} -1.44 \\ 0.66 \\ -0.01 \\ 0.85 \end{bmatrix}$$

Similar type of behaviour can be observed if other parameters are varied

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ p \\ y^* \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.44 \\ 0.096 \\ p \\ 5.0 \end{bmatrix}$$

and let $p \in [\, 0.0,\ 0.032\,]$ be the bifurcation parameter.

Fig. 5 below shows the results for the parameter $p \in [\, 0.001, 0.032\,]$.
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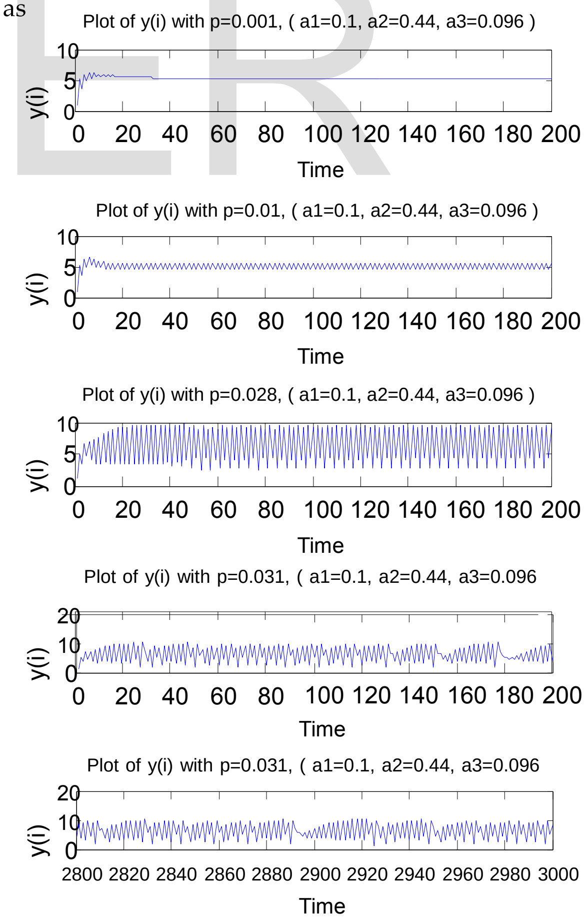

Fig. 5 - Plot of $y(i)$

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Again from the simulation results in Fig. 5, the system going through various bifurcations, from stable to limit cycle to period doubling through to chaotic behaviour as parameter p gradually increases.

6.2 Third Order Plant with Second Order Model

It is now possible to assume a model which is second order and thus a better approximation to our third order plant. It is expected that system to be more structurally stable as parameter values are varied than our first order model assumption. The corresponding set of first order equations become:

$$w_1(t) = w_2(t-1)$$
$$w_2(t) = y(t-1)$$
$$\hat{z}(t) = \hat{a}_2(t-1)$$

$$y(t) = a_1 y(t-1) + a_2 w_2(t-1) + a_3 w_1(t-1) + y^* - \hat{a}_1(t-1)y(t-1) - \hat{z}(t-1)w_2(t-1)$$

$$\hat{a}_1(t) = \hat{a}_1(t-1) + py(t-1)[\, y(t) - y^* \,]$$
$$\hat{a}_2(t) = \hat{a}_2(t-1) + pw_2(t-1)[\, y(t) - y^*]$$

Since we have a single reference input value $y(t) = y^*$ and we are estimating two parameters $\hat{a}_1$, $\hat{a}_2$ then both $\hat{a}_1$ and $\hat{a}_2$ cannot be determined explicitly and so we do not have a unique equilibrium point. We can still simulate our system equations above to see our system behaviour for different parameter values.

Letting parameter a_3 vary over a range of values from 0.1 to 0.63, with

$$\begin{bmatrix} w_1(0) \\ w_2(0) \\ y(0) \\ \hat{z}(0) \\ \hat{a}_1(0) \\ \hat{a}_2(0) \end{bmatrix} = \begin{bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ p \\ y^* \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.8 \\ a_3 \\ 0.005 \\ 5.0 \end{bmatrix}$$

Fig. 6 shows $y(i)$ as a_3 ranges from 0.1 – 0.63.

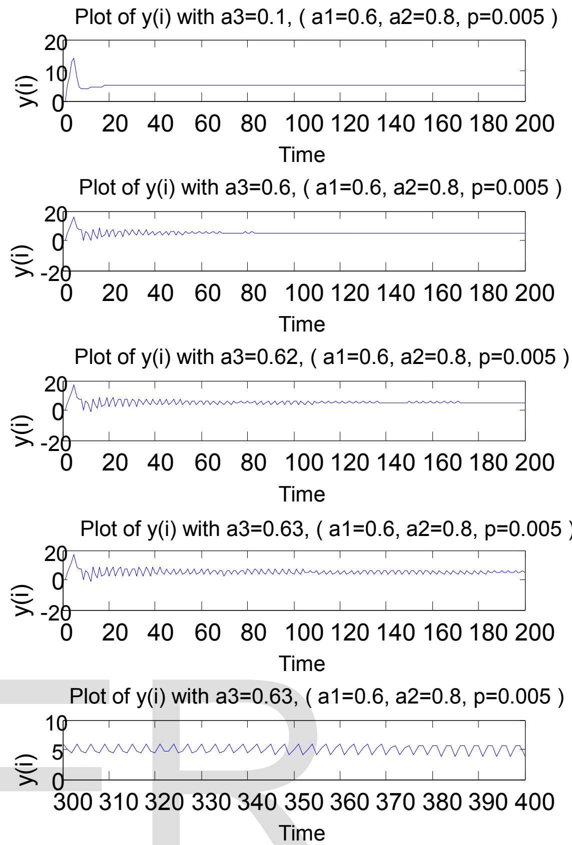

Fig. 6 - Plot of $y(i)$

7 Discussions and Conclusions

It is seen that model reference adaptive systems which can be represented as nonlinear discrete maps undergo various bifurcations before the system becomes unstable and eventually unbounded. Before the system becomes unbounded it traverses a region of increasingly complex dynamics characterised by random unpredictable behaviour which is termed as chaotic. A similar type of behaviour was observed for different bifurcation parameter values. A discrete time adaptive control system which had shown to display chaotic behaviour was extended into a higher order system. It was shown that the higher order system was also likely to display chaotic behaviour under certain parameter conditions. Observations of chaotic behaviour for higher order plants with first order model assumptions were expected because these higher order systems can be reduced back into second and third order systems by choosing the appropriate parameter values. A generalised analytical form for the characteristic equation was determined and this allowed the parameter values to be determined for which the system just becomes unstable. However,

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ISSN 2229-5518

to find the values of the bifurcation parameter for which chaotic behaviour occurs simulation was used.

When a second order model assumption to the third order plant was used, the system was found to be stable for a larger range of parameter values. It was seen in section 6.2 as the parameter a_3 was increased beyond a critical point the system no longer converged to the desired value but began to diverge slowly and gradually became unbounded. Chaotic behaviour was not observed as in the previous section, and a possible suggestion for this could be that the system does not have unique equilibrium points as was the case for the earlier systems we studied with a first order model assumption. The chaotic phenomena was generally associated with the equilibrium point continually bifurcating from a stable fixed point to a limit cycle and through continuous period doubling and eventually leading to complex chaotic behaviour.

From a control aspect, it is important to know the dynamical behaviour of a system for different parameter values. It is reasonable to assume that local stability can be maintained if the period doubling phenomena can be avoided. Knowledge of the overall system behaviour for different parameter values is crucial in attaining any desired type of behaviour or maybe avoids an undesirable type of behaviour.

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